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LETTER TO THE EDITOR

Lattice shapes and scaling functions for bond random percolation on honeycomb lattices

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Abstract. We use a histogram Monte Carlo simulation method to calculate the scaling functions of the existence probability E_p and the percolation probability P of the bond random percolation model on honeycomb lattices with aspect ratios of 0.5 and 2.0. We find that such different aspect ratios give quite different scaling functions near the critical region. However, they give a consistent critical point, critical exponents, and the thermodynamic order parameter from renormalization-group calculations. Some interesting theoretical problems related to this work are discussed.

Finite-size scaling has been an important subject of research in recent decades [1–6]. According to the theory of finite-size scaling [1–6] if the dependence of a physical quantity Q of a thermodynamic system on the parameter t, which vanishes at the critical point, may be written as $Q(t) \sim t^a$ near the critical point, then for a finite system of linear dimension L at t, the corresponding quantity $Q(L, t)$ may be written as

$$Q(L,t) \sim L^{-ay_t} F(tL^{y_t}) \tag{1}$$

where y_t $(= \nu^{-1})$ is the thermal scaling power and $F(x)$, $x = tL^{y_t}$, is called a scaling function. When finite-size scaling is valid, the scaled data $Q(L,t)/L^{-ay_t}$ for different values of L fall on the same curve, represented by $F(x)$, if they are plotted as a function of the scaling variable x. Thus it is important to develop new methods to calculate scaling functions and to know the behaviour of the scaling functions under various conditions.

Recently, Hu developed the histogram Monte Carlo simulation method (HMCSM) [7–9] which may be easily applied to calculate the scaling functions for phase transition and percolation models [10–14]. In a recent letter [13], Hu used the HMCSM to calculate the scaling functions of the existence probability E_p and the percolation probability P of the site percolation model on square lattices with free and periodic boundary conditions. He found that different boundary conditions give quite different scaling functions near the critical region. However, they give a consistent critical point, critical exponents, and the thermodynamic order parameter from renormalization-group calculations. In this letter, we use the HMCSM to calculate the existence probability E_p and the percolation probability P of the bond random percolation model on the honeycomb lattices with different aspect ratios and linear dimensions, where the aspect ratio is the ratio of the horizontal linear dimension to the vertical linear dimensions of the lattice. For a given aspect ratio, the calculated E_p

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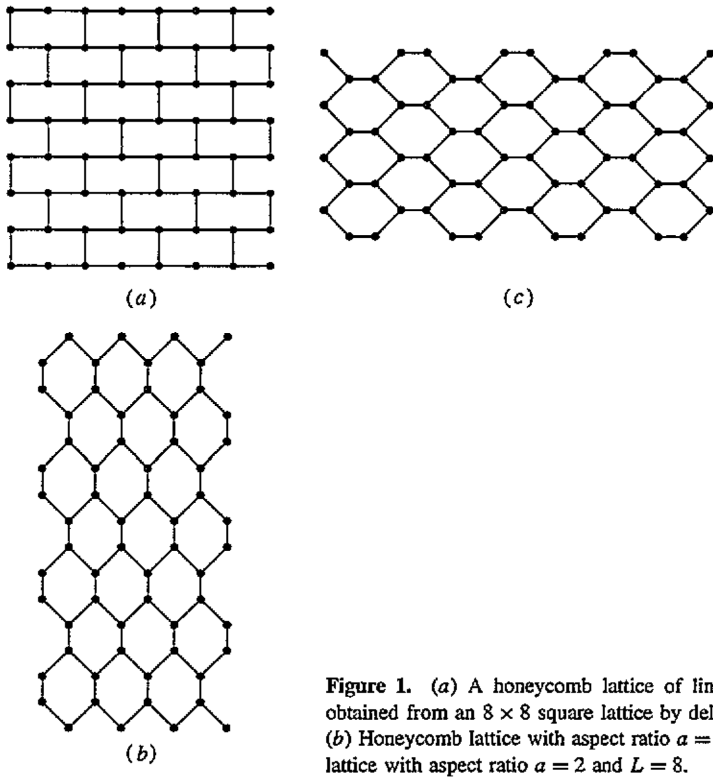

Figure 1. (*a*) A honeycomb lattice of linear dimension $L = 8$ may be obtained from an 8×8 square lattice by deleting half of the vertical bonds. (*b*) Honeycomb lattice with aspect ratio $a = 0.5$ and $L = 8$. (*c*) Honeycomb lattice with aspect ratio $a = 2$ and $L = 8$.

and P have very good scaling behaviour. We find that the scaling functions for different aspect ratios are quite different near the critical region. However, when we apply the large cell-to-cell Monte Carlo renormalization-group method [7–9] to calculate the critical point, critical exponents, and the thermodynamic order parameter, we find that lattices of different aspect ratios give consistent results. The implications of our calculated results on some theoretical problems of current interest will be discussed at the end of this letter.

The honeycomb lattice may be generated from a square lattice by removing one quarter of the nearest-neighbour bonds as shown in figure 1. Figure 1(*a*) is originally an 8×8 square lattice. By removing one half of the vertical bonds of the original square lattice, we obtain figure 1(*a*), whose site-bond connections are the same as the honeycomb lattice shown in figure 1(*b*). For convenience, we call figure 1(*b*) a honeycomb lattice of linear dimension 8, whose number of lattice sites N is 64 and whose number of nearest-neighbour bonds E is 84 for the free boundary condition and is 96 for the periodic boundary condition. In general, for a honeycomb lattice G of linear dimension L, the number of lattice sites N is L^2 and the number of nearest-neighbour bonds is $L \times (L-1) \times \frac{3}{2}$ for the free boundary condition and is $L \times L \times \frac{3}{2}$ for the periodic boundary condition. Rotating figure 1(*b*) by 90°, we obtain figure 1(*c*) whose number of lattice sites N and number of nearest-neighbour bonds E are the same as those of figure 1(*b*). In this letter, we consider the bond random percolation model (BRPM) on lattices G of N sites and E nearest-neighbour bonds shown in figures 1(*b*) and (*c*), but with linear dimensions L being 8, 16, 32 and 64.

In the BRPM on G, each bond of G is occupied with a probability p, where $0 \leqslant p \leqslant 1$. The probability weight for the appearance of a subgraph G' of $b(G')$ occupied bonds is given by

$$\pi(G', p) = p^{b(G')}(1-p)^{E-b(G')}. \tag{2}$$

In this letter, the cluster which extends from the top line of G to the bottom line of G is called a percolating cluster. The subgraph whose largest cluster is percolating is called a percolating subgraph and will be denoted by G'_p. The subgraph whose largest cluster is not percolating is called a non-percolating subgraph and will be denoted by G'_f. The existence probability $E_p(G, p)$ and the percolation probability $P(G, p)$ for the BRPM on G are given by

$$E_p(G, p) = \sum_{G'_p \subseteq G} \pi(G'_p, p) \tag{3}$$

$$P(G, p) = \sum_{G'_p \subseteq G} \pi(G'_p, p) N^*(G'_p)/N \tag{4}$$

where $\pi(G'_p, p)$ is defined by (2). The sums in (3) and (4) are over all G'_p of G; $N^*(G')$ is the total number of lattice sites in the largest cluster of G. The definitions of G_p, G_f, and $N^*(G')$ in the present letter are different from the corresponding definitions of [7, 8]. The new definitions allow us to save a lot of computing time and therefore we may do the calculations for larger systems.

In the histogram Monte Carlo simulation of the BRPM on G [7], we choose a sequence of bond probabilities of increasing magnitudes: $0 < p_1 < p_2 < p_3 < \cdots < p_w < 1$. For a given p_j, $1 \leqslant j \leqslant w$, we generate N_R different subgraphs G'. The data obtained from wN_R different G' are then used to construct three arrays of length E with elements $N_p(b)$, $N_f(b)$, and $N_{pp}(b)$, $0 \leqslant b \leqslant E$, which are, respectively, the total numbers of generated percolating subgraphs with b occupied bonds, the total number of generated non-percolating subgraphs with b occupied bonds, and the sum of $N^*(G')$ over subgraphs with b occupied bonds. The existence probability E_p and the percolation probability P at any value of the bond occupation probability p may be calculated from the following equations [8]:

$$E_p(G, p) = \sum_{b=0}^{E} p^b (1-p)^{E-b} C_b^E \frac{N_p(b)}{N_p(b) + N_f(b)} \tag{5}$$

$$P(G, p) = \sum_{b=0}^{E} p^b (1-p)^{E-b} C_b^E \frac{N_{pp}(b)}{N_p(b) + N_f(b)}. \tag{6}$$

We have used (5) and (6) to calculate the existence probability $E_p(G, p)$ and the percolation probability $P(G, p)$ of the BRPM on the honeycomb lattices with the free boundary condition and with linear dimensions $L = 8$, 16, 32 and 64. We consider both the aspect ratio $a = 0.5$ and 2. Typical calculated results of E_p and P are shown in figure 2. For the BRPM on the honeycomb lattice, it is generally believed that the *exact* y_t, y_h and p_c are 0.75, 1.8958,..., and 0.65271, respectively [6]. Using the exact value of y_t and p_c [6], we have plotted the data for $E_p(G, p, q)$ represented in figure 2(*a*) as a function of $x = (p - p_c)L^{y_t}$ in figure 3(*a*). Since the critical exponent of E_p is zero [6], we need not divide E_p by the factor $L^{-\alpha y_t}$. Using the same values of y_t and p_c, we have also plotted $P(G, p, q)/L^{-\beta y_t}$ for $P(G, p)$ presented in figure 2(*b*) as a function of $x = (p - p_c)L^{y_t}$ in figure 3(*b*). Figures 3(*a*) and (*b*) show that E_p and P have nice finite-size scaling behaviour. However, the scaling functions for $a = 0.5$ and $a = 2$ are quite different. As L approaches large values, $E_p(G, p_c)$ approaches 0.25(6) for the case $a = 0.5$ and approaches 0.75(7) for the case $a = 2.0$.

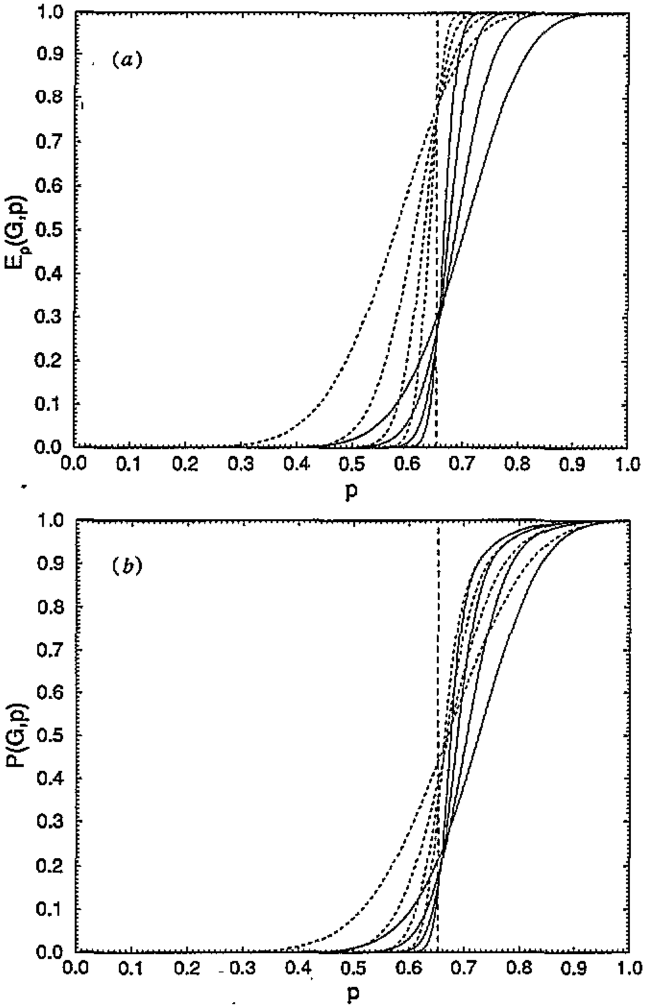

Figure 2. The calculated results for the bond random percolation model on the honeycomb lattices with linear dimensions L: 8, 16, 32 and 64. The w and N_R values are 301 and 2×10^5, respectively. Full curves are for $a = 0.5$ and broken curves are for $a = 2$. (*a*) E_p as a function of p. (*b*) P as a function of p.

We have used the percolation renormalization-group equations [7] to calculate the critical point p_c, the thermal scaling power y_t, and the field scaling power y_h for the BRPM on the honeycomb lattices with $a = 0.5$ and 2. In both cases, we use $w = 301$ and $N_R = 200\,000$ for $L_1 = 64$ and for $L_2 = 32$. We obtain $p_c = 0.65(4)$, $y_t = 0.7(5)$, and $y_h = 1.89(9)$ for $a = 0.5$ and $p_c = 0.65(3)$, $y_t = 0.7(5)$, and $y_h = 1.89(6)$ for $a = 2$. Two results are consistent. Our numerical results are very close to exact results [6].

We have used method of [9] to calculate the thermodynamic order parameter P_∞ for the BRPM with $a = 0.5$ and 2, which are shown by full and broken curves in figure 4, respectively. The two calculated results are consistent.

From our calculated results, we may discuss some theoretical problems of current interest. The results of this letter show that the scaling functions of percolation problems depends on the aspect ratio a. Using the idea of Langlanda *et al* [15], we may adjust the

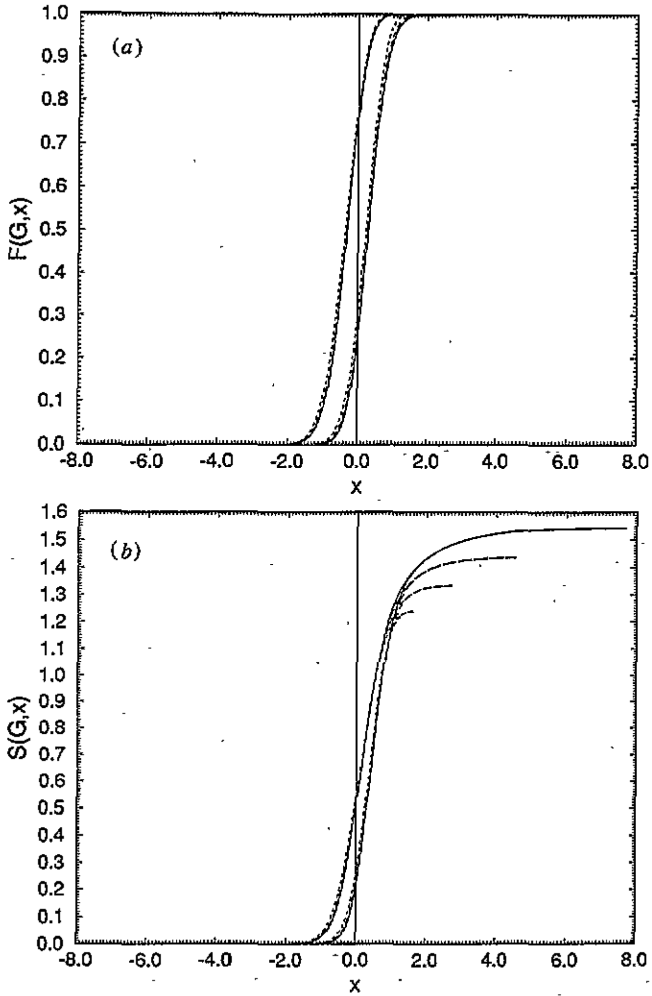

Figure 3. Scaling functions for the honeycomb lattices with linear dimensions $L = 8$ (dotted curves), 16 (broken curves), 32 (long broken curves), and 64 (full curves). The lower and upper lines are for a being 0.5 and 2, respectively. (a) The calculated E_p as a function of x, where $x = (p - p_c)L^{y_t}$. The function is the scaling function $F(G, x)$. (b) The calculated $P/L^{-\beta y_t}$ as a function of x, where $x = (p - p_c)L^{-y_t}$. The function is the scaling function $S(G, x)$.

ratio a for various lattices on the same space dimensions so that different lattices give the same value of $E_p(G, p_c)$, i.e. $F(G, 0)$. Under such conditions, we may ask whether the scaling functions, $F(G, x)$, for various lattices are consistent for x near 0. If this is the case, then we have the *universal scaling function* considered by Privman and Fisher [3]. We are using the histogram Monte Carlo simulation method [7, 8] to calculate such a ratios and the scaling functions for percolation on various two to five-dimensional lattices.

In summary, the scaling functions of E_p and P at near p_c depends sensitively on aspect ratio of the lattice. The histogram Monte Carlo simulation method [7–9] is useful for identifying the *universality classes* of $E_p(G, p_c)$, which is of much current interest [16–18], and for obtaining the scaling functions for E_p and P, and for calculating the critical point, critical exponents, and the thermodynamic order parameter.

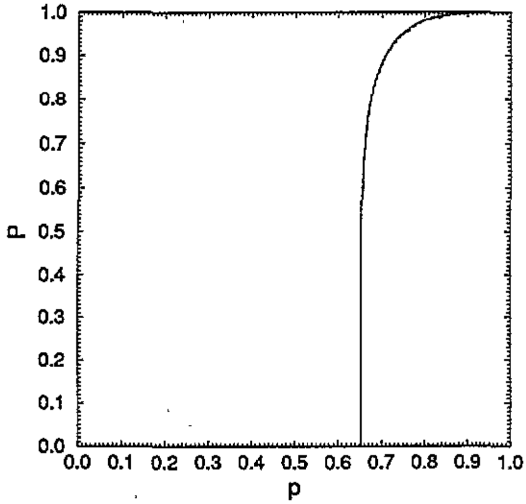

Figure 4. The calculated thermodynamic order parameters $P_\infty(G, p)$ of the bond random percolation model on the honeycomb lattice. The full and broken curves are for the aspect ratio a being 0.5 and 2, respectively.

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